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In this paper, we propose a new method, based on Bezoutian matrices, for computing a nontrivial multiple of the resultant over a projective variety X, which is described on an open subset by a parameterization. This construction, which generalizes the classical and toric one, also applies for instance to blowing up varieties and to residual intersection problems. We recall the classical notion of resultant over a variety X. Then we extend it to varieties which are parameterized on a dense open subset and give new conditions for the existence of the resultant over these varieties. We prove that any maximal nonzero minor of the corresponding Bezoutian matrix yields a nontrivial multiple of the resultant. We end with some experiments.

1. Introduction

Resultant theory has a long mathematical story, starting with the resolution of linear systems. The first explicit construction of the so-called resultant of two univariate polynomials, proposed by E. Bézout and L. Euler (Bézout, 1764), was followed by the well known dialytic method of Sylvester (Sylvester, 1840). Generalizations to multivariate polynomials appeared soon after (Sylvester, 1841), becoming an intensive subject of study (Salmon, 1885), (Macaulay, 1902), (Dixon, 1908), (Van der Waerden, 1948) ...

After the dark period "Il faut éliminer l'élimination", these last decades have witnessed a renewal of elimination theory (Jouanolou, 1991), (Gelfand *et al.*, 1994), (Eisenbud, 1994), partly motivated by applications in effective algebraic geometry and more specially in polynomial system solving (Chistov, 1986), (Grigoryev, 1986), (Pedersen and Sturmfels, 1993), (Manocha and Canny, 1993), (Chardin, 1995), (Mourrain, 1998). Indeed many operations used in this domain involve projections of varieties and elimination of variables. Resultant constructions yield a direct answer to such problems. After a preprocessing step of the polynomial equations, one obtain the "eliminant" polynomial by specialization of the input coefficients in the determinant of the constructed matrix. This

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approach is particularly interesting for numerical solver (based on eigenvalue computations), because it provides a template construction, which applies for a large class of input systems (Manocha, 1994), (Bondyfalat *et al.*, 1998).

However such methods suffer from a problem of genericity, when the input system yields a degenerate resultant construction. This problem has lead to new developments, extending the notion of resultant to more general varieties than the projective space (Gelfand *et al.*, 1994). The recent efforts in this direction concern resultants over toric varieties, and more precisely explicit matrix constructions whose determinant is a nontrivial multiple of the toric resultant (Sturmfels, 1993), (Gelfand *et al.*, 1994)[chap. 8], (Canny and Emiris, 1993).

In this work, we aim at extending such constructions to resultants over general varieties. We propose a systematic method based on Bezoutian matrices, which yields a nontrivial multiple of the resultant over a projective variety X, when a dense open subset of this variety can be parameterized. It generalizes the classical and toric one, corresponding to varieties parameterized by monomial maps, and it also applies to blowing up varieties or residual intersection problems.

We divide our presentation as follows. In the next section, we recall the classical notion of resultant over a variety X, giving some conditions for which this resultant is well defined. These conditions are essentially those given in (Gelfand *et al.*, 1994), but reformulated in simpler terms. In subsection 2.2, we extend this approach to varieties which are described by a parameterization on an open subset and we give new conditions (less restrictive) for the existence of the resultant over these varieties. In section 3, we recall the definition and a fundamental property of multivariate Bezoutian matrices. We prove that any maximal nonzero minor of these matrices yields a nontrivial multiple of the resultant. Finally, we illustrate this construction by 3 examples.

Before going into details, here are the notations that will be used hereafter. Let $\mathbb{K}$ be a field and $R = \mathbb{K}[t_1, \ldots, t_n] = \mathbb{K}[t]$ be the ring of polynomials in the variables $t_1, \ldots, t_n$, with coefficients in $\mathbb{K}$. For $\alpha = (\alpha_1, \ldots \alpha_n) \in \mathbb{N}^n$, $t^\alpha = t_1^{\alpha_1} \ldots t_n^{\alpha_n}$. We denote by $\overline{\mathbb{K}}$ the algebraic closure of $\mathbb{K}$, $\mathbb{P}^n$ (resp. $\mathbb{A}^n$) the projective (resp. affine) space over $\overline{\mathbb{K}}$.

Introducing new variables $z = (z_1, \ldots, z_n)$, we will identify the algebra $R \otimes_{\mathbb{K}} R$ with $\mathbb{K}[t, z] = \mathbb{K}[t_1, \ldots, t_n, z_1, \ldots, z_n]$.

If I is an ideal of R, we denote by $\mathcal{V}_{\overline{\mathbb{K}}}(I)$ (or simply $\mathcal{V}(I)$) the set of common roots in $\overline{\mathbb{K}}^n$ of elements of I. More generally, we denote by $\mathcal{V}_X(I)$ the set of common roots of (resp. homogeneous) elements of I in the algebraic (resp. projective) variety X. The quotient ring of R by the ideal I is denoted by $\mathcal{A} = R/I$. The class of a polynomial $p \in R$ in $\mathcal{A}$ is denoted by $\overline{p}$.

We denote by $\widehat{R}$ the dual of R (i.e. the set of linear forms from R to $\mathbb{K}$), $\widehat{\mathcal{A}}$ the vector space of $\mathbb{K}$-linear maps from $\mathcal{A}$ to $\mathbb{K}$. We will identify $\widehat{\mathcal{A}}$ with $I^\perp = \{\Lambda \in \widehat{R} : \Lambda(f) = 0 \text{ for all } f \in I\}$. It has a natural structure of $\mathcal{A}$-module: $\forall \Lambda \in \widehat{\mathcal{A}}, \forall a \in \mathcal{A}$, $a \cdot \Lambda : b \in \mathcal{A} \mapsto \Lambda(a\, b) \in \mathbb{K}$.

2. Resultant theory

Elimination theory deals with the problem of finding conditions on parameters of a system of equations, so that these equations have a common solution in a fixed algebraic variety X.

2.1. Classical resultant case

The typical situation is the case of $n+1$ "polynomials"

$$\mathbf{f_c} := \begin{cases} f_0(x) &= \sum_{j=0}^{k_0} c_{0,j}\,\psi_{0,j}(x) \\ \quad\vdots & \\ f_n(x) &= \sum_{j=0}^{k_n} c_{n,j}\,\psi_{n,j}(x) \end{cases}$$

where $\mathbf{c} = (c_{i,j})$ are parameters, x is a point of the variety X of dimension n, and the vector $\mathcal{L}_i = (\psi_{i,j})_{j=0,\ldots,k_i}$ is a regular map from X to $\mathbb{P}^{k_i}$ (see (Harris, 1992)) independent of $\mathbf{c}$. In the language of modern algebraic geometry, the $\mathcal{L}_i$ correspond to line bundles on X and the f_i to generic global sections (see (Gelfand *et al.*, 1994)).

The elimination problem consists, in this case, in finding necessary (and sufficient) conditions on $\mathbf{c}$ such that the system $\mathbf{f_c} = 0$ has a solution in X.

In the classical case, X is the projective space $\mathbb{P}^n$, $\mathcal{L}_i$ is the vector of all monomials of a fixed degree d_i, and the function f_i is a generic homogeneous polynomial of degree d_i. The necessary and sufficient condition on $\mathbf{c}$ such that the homogeneous polynomials $f_0, \ldots, f_n$ have a common root in $\mathbb{P}^n$ is $\mathrm{Res}_{\mathbb{P}^n}(\mathbf{f_c}) = 0$, where $\mathrm{Res}_{\mathbb{P}^n}(\mathbf{f_c})$ is the *classical projective resultant* (see (Macaulay, 1902), (Van der Waerden, 1948)).

Considering a geometric point of view: We look for the values of parameters $\mathbf{c} = (c_{i,j})$ such that there exists $x \in X$ with $\sum_{j=0}^{k_i} c_{i,j}\psi_{i,j}(x) = 0$ for $i = 0, \ldots, n$. In other words, the vector $\mathbf{c}$ is the projection of the point $(\mathbf{c}, x)$ of the *incidence variety*

$$W_X = \{(\mathbf{c}, x) \in \mathbb{P}^{k_0} \times \cdots \times \mathbb{P}^{k_n} \times X : \sum_{j=0}^{k_i} c_{i,j}\psi_{i,j}(x) = 0, i = 0, \ldots, n\}.$$

We denote by $\pi_1 : W_X \to \mathbb{P}^{k_0} \times \cdots \times \mathbb{P}^{k_n}$ and $\pi_2 : W_X \to X$ the two natural projections. The image of W_X by π_1 is precisely the set of values of parameters $\mathbf{c}$ for which the system has a root (in X). The image by π_2 of a point of W_X is a solution in X of the associated system.

Definition 2.1. *If $\pi_1(W_X)$ is an irreducible hypersurface, then "its" equation (unique up to a scalar) will be called the resultant of $f_0, \ldots, f_n$. It will be denoted by* $\mathrm{Res}_X(\mathbf{f_c})$.

In order to be in this case (i.e. $\pi_1(W_X)$ is an irreducible hypersurface), we impose the following conditions:

$$(\mathbf{C}) \begin{cases} \textbf{(C1)} & X \text{ is a projective irreducible variety of dimension } n. \\ \textbf{(C2)} & \text{For each } i = 0, \ldots, n,\ \mathcal{L}_i \text{ is a regular map from } X \text{ to } \mathbb{P}^{k_i}. \\ \textbf{(C3)} & \text{For generic values of } \mathbf{c}, \text{ the system } \mathbf{f_c} = 0 \text{ has no solution in } X. \end{cases}$$

The condition **(C1)** is required, because affine algebraic varieties do not behave correctly by projection, but projective ones do. The irreducibility of X is not necessary, but it simplifies the presentation. By decomposing the variety X into irreducible components, we can reduce to this case. **(C2)** will give us the properties of W_X. **(C3)** is obviously needed, if we want to define a resultant polynomial.

Theorem 2.2. *If the conditions* **(C)** *are satisfied, the projection $Z = \pi_1(W_X)$ is an irreducible hypersurface of $\mathbb{P}^{k_0} \times \cdots \times \mathbb{P}^{k_n}$.*

Proof. Consider a point $x \in X$. Its fiber $\pi_2^{-1}(x)$ is a linear space of $\mathbb{P}^{k_0} \times \cdots \times \mathbb{P}^{k_n} \times \{x\}$. By condition **(C2)**, this space is of dimension $\sum_{i=0}^{n} k_i - n - 1$. From the fiber theorem (see (Shafarevitch, 1974)[p. 60] or (Harris, 1992)[p. 139]), we deduce that W_X is irreducible and of dimension $\sum_{i=0}^{n} k_i - 1$. Thus, its projection Z by π_1 is an irreducible variety of dimension $\leq \sum_{i=0}^{n} k_i - 1$.

Let U be the dense subset of $\mathbb{P}^{k_0} \times \cdots \times \mathbb{P}^{k_n}$ such that $\mathcal{V}_X(f_0, \ldots, f_n)$ is empty (in X). Let U' be the subset of $\mathbb{P}^{k_0} \times \cdots \times \mathbb{P}^{k_n}$ such that $\mathcal{V}_X(f_1, \ldots, f_n)$ is finite. We have $U \subset U'$, for if $\mathcal{V}_X(f_1, \ldots, f_n)$ is not zerodimensional (i.e. it is of dimension ≥ 1) then $\mathcal{V}_X(f_0, \ldots, f_n) = \mathcal{V}_X(f_1, \ldots, f_n) \cap \mathcal{V}_X(f_0)$ is not empty. Therefore U' is a dense open subset of $\mathbb{P}^{k_0} \times \cdots \times \mathbb{P}^{k_n}$ which implies that $W_X \cap (U' \times X)$ is a dense subset of W_X and projects by π_1 onto $Z \cap U'$. As $\mathcal{V}_X(f_1, \ldots, f_n)$ is finite for any $\mathbf{c} \in Z \cap U'$, $\pi_1^{-1}(\mathbf{c}) = \{(\mathbf{c}, \zeta) \ : \zeta \in \mathcal{V}_X(f_1, \ldots, f_n) \cap \mathcal{V}_X(f_0)\}$ is finite for $\mathbf{c} \in U'$. Therefore, W_X and Z are of the same dimension and Z is an hypersurface of $\mathbb{P}^{k_0} \times \cdots \times \mathbb{P}^{k_n}$. □

2.2. Resultant over a parameterized variety

We consider here systems of the form

$$\mathbf{f_c} := \begin{cases} f_0(t) &= \sum_{j=0}^{k_0} c_{0,j}\, \kappa_{0,j}(t) \\ \quad\vdots & \\ f_n(t) &= \sum_{j=0}^{k_n} c_{n,j}\, \kappa_{n,j}(t) \end{cases} \tag{2.1}$$

where $t = (t_1, \ldots, t_n)$ and the $\kappa_{i,j}(t)$ are polynomials. We assume that they are not zero, otherwise we drop them from the decomposition (2.1). Let $\mathcal{K}_i = (\kappa_{i,j})_{j=0,\ldots,k_i}$ be the vector of polynomials defining f_i. We are looking for conditions on the coefficients $\mathbf{c} = (c_{i,j})$ such that there exists $t \in \mathbb{A}^n$ with $\mathcal{K}_i(t) \neq 0$ and $f_0(t) = \cdots = f_n(t) = 0$.

Let U be the open subset of $\mathbb{A}^n$ such that $\mathcal{K}_i(t) \neq 0$, for $i = 0, \ldots, n$. Let $\sigma_0(t), \ldots, \sigma_N(t)$ be polynomials in R defining a map

$$\begin{aligned} \sigma : U &\rightarrow \mathbb{P}^N \\ t &\mapsto \big(\sigma_0(t) : \cdots : \sigma_N(t)\big), \end{aligned}$$

and $\psi_{i,j}(x_0, \ldots, x_N)$, $i = 0, \ldots, n$, $j = 0, \ldots, k_i$ be homogeneous polynomials, such that

$$\kappa_{i,j}(t) = \psi_{i,j}\big(\sigma_0(t), \ldots, \sigma_N(t)\big) \text{ and } \deg(\psi_{i,j}) = \deg(\psi_{i,0}) \geq 1.$$

Notice that the maps σ and $\psi_{i,j}$ are not uniquely defined. We may have many choices for these functions. See section 4 for examples.

Let X^o be the image of σ and $X = \overline{X}^o$ the closure of X^o in $\mathbb{P}^N$. In order to be able to construct the resultant associated to the system (2.1) on the variety X, we make two hypotheses:

$$(\mathbf{D}) \begin{cases} (\mathbf{D1}) & \text{The Jacobian matrix of } \sigma = (\sigma_i)_{i=0,\ldots,N} \text{ is of rank } n \text{ at one point of } U. \\ (\mathbf{D2}) & \text{For generic } \mathbf{c},\ f_1 = \cdots = f_n = 0 \text{ has a finite number of solutions in } U. \end{cases}$$

When the field $\mathbb{K}$ has characteristic 0, **(D1)** is equivalent to the assumption that X has dimension n. We will show that these conditions are sufficient to define the resultant. Let $U^o = \{t \in U : \kappa_{i,0}(t) \neq 0, \text{for } i = 0, \ldots, n\}$ be the dense open subset of U and consider the parameterization

$$\begin{aligned} \tau : \mathbb{P}^{k_0 - 1} \times \cdots \times \mathbb{P}^{k_n - 1} \times U^o &\rightarrow \mathbb{P}^{k_0} \times \cdots \times \mathbb{P}^{k_n} \times \mathbb{P}^N \\ (\tilde{\mathbf{c}}_0, \ldots, \tilde{\mathbf{c}}_n, t) &\mapsto \big(\mathbf{c}_0, \ldots, \mathbf{c}_n, \sigma(t)\big) \end{aligned}$$

with $\mathbf{c}_i = (c_{i,0}, \tilde{\mathbf{c}}_i)$ and $c_{i,0} = -\frac{1}{\kappa_{i,0}(t)} \sum_{j=1}^{k_i} c_{i,j}\kappa_{i,j}(t)$ ($\kappa_{i,0}(t) \neq 0$). We denote by W^o the image of this map, $W = \overline{W}^o$ its closure in $\mathbb{P}^{k_0} \times \cdots \times \mathbb{P}^{k_n} \times \mathbb{P}^N$, $\pi_1 : \mathbb{P}^{k_0} \times \cdots \times \mathbb{P}^{k_n} \times \mathbb{P}^N \to \mathbb{P}^{k_0} \times \cdots \times \mathbb{P}^{k_n}$, and $\pi_2 : \mathbb{P}^{k_0} \times \cdots \times \mathbb{P}^{k_n} \times \mathbb{P}^N \to \mathbb{P}^N$ the canonical projections.

THEOREM 2.3. *Under the conditions* **(D)***, the variety W is irreducible and projects onto a hypersurface $Z = \pi_1(W)$. It is defined by one equation* $\mathrm{Res}_X(\mathbf{f_c}) = 0$*, where* $\mathrm{Res}_X(\mathbf{f_c})$ *is the resultant of* $\mathbf{f_c}$ *on the parameterized variety X.*

PROOF. The variety W is the closure of a parameterized variety W^o. Therefore, it is irreducible and its projection Z is also irreducible.

According to condition **(D1)**, the Jacobian of σ is of rank n at one point of U and thus on an open subset of U, which implies that the variety X^o (and thus X) is of dimension n.

The fibers of the projection $\pi_2 : W^o \to X^o$ are linear spaces of dimension $\sum_{i=0}^n k_i - n - 1$, for we have $\mathcal{K}_i(t) \neq 0$ when $t \in U$. By the fiber theorem ((Shafarevitch, 1974)[p. 60] or (Harris, 1992)[p. 139]), we deduce that W is of dimension $\sum_{i=0}^n k_i - 1$.

Consider now the restriction of the projection π_1 to $W^o : W^o \to \mathbb{P}^{k_0} \times \cdots \times \mathbb{P}^{k_n}$. According to condition **(D2)**, there exists an open subset of $\mathbb{P}^{k_0} \times \cdots \times \mathbb{P}^{k_n}$ on which the number of solutions of $f_1 = \cdots = f_n = 0$ is finite. The fibers of π_1 on this open subset is therefore of dimension 0. This shows that the projection $\pi_1(W^o)$ and thus Z is of the same dimension as W, that is a hypersurface of $\mathbb{P}^{k_0} \times \cdots \times \mathbb{P}^{k_n}$, defined (up to a scalar) by one equation. □

COROLLARY 2.4. *For any specialization of the parameters* $\mathbf{c} = (c_{i,j})$, $\mathrm{Res}_X(\mathbf{f_c}) = 0$ *if and only if there exists* $(\mathbf{c}, x) \in W$ *such that* $\tilde{f}_i(x) := \sum_{j=0}^{k_i} c_{i,j}\, \psi_{i,j}(x) = 0$*, for* $i = 0, \ldots, n$.

PROOF. As the fibers of π_2 above X^o are of dimension $\sum_{i=0}^n k_i - n - 1$ and W is of dimension $\sum_{i=0}^n k_i - 1$, the image $\pi_2(W)$ is an irreducible variety of dimension n, containing X^o. This shows that $X = \pi_2(W)$.

Consequently $\mathrm{Res}_X(\mathbf{f_c}) = 0$ iff there exists $x \in X$ such that $(\mathbf{c}, x) \in W$, i.e. satisfying $\tilde{f}_i(x) = 0$, $i = 0, \ldots, n$. □

REMARK 2.5. The degree of the resultant $\mathrm{Res}_X(\mathbf{f_c})$ in the coefficients $c_{i,j}$ of f_i is bounded by (but not necessarily equal to) the generic number of roots $V_i = \mathcal{V}_X(\tilde{f}_0, \ldots, \tilde{f}_{i-1}, \tilde{f}_{i+1}, \ldots, \tilde{f}_n)$. In the case where the linear forms (in $c_{i,j}$) $\tilde{f}_i(\zeta)$, $\zeta \in V_i$, are all distinct, then the degree of $\mathrm{Res}_X(\mathbf{f_c})$ in $c_{i,j}$ is exactly the number of generic roots. This is the case when the line bundle $\mathcal{L}_i$ is very ample or when $t_1, \ldots, t_n$ appear among the $\kappa_{i,j}$, $j = 0, \ldots, k_i$, as it is illustrated in section 4.1 and 4.3.

3. Bezoutians

In this section, we relate Bezoutians and Resultants. We show that in the case (of practical importance) where an open subset of the projective variety X is parameterized by a polynomial map, the resultant is a factor of any maximal minor of the Bezoutian matrix. See also (Kapur *et al.*, 1994), (Cardinal and Mourrain, 1996), (Elkadi and Mourrain, 1999b), (Elkadi and Mourrain, 1999a) for connected results.

3.1. Definitions and properties

We recall the construction of Bezoutian matrices, that we will use hereafter.

Definition 3.1. *The Bezoutian $\Theta_{f_0,\dots,f_n}$ of the polynomials $f_0,\dots,f_n \in R$ (or simply Θ_{f_0} if $f_1,\dots,f_n$ are fixed) is the polynomial in $R\otimes_{\mathbb{K}} R$ defined by*

$$\Theta_{f_0,\dots,f_n}(t,z) := \begin{vmatrix} f_0(t) & \theta_1(f_0)(t,z) & \cdots & \theta_n(f_0)(t,z) \\ \vdots & \vdots & \vdots & \vdots \\ f_n(t) & \theta_1(f_n)(t,z) & \cdots & \theta_n(f_n)(t,z) \end{vmatrix},$$

where

$$\theta_i(f_j)(t,z) := \frac{f_j(z_1,\dots,z_{i-1},t_i,\dots,t_n) - f_j(z_1,\dots,z_i,t_{i+1},\dots,t_n)}{t_i - z_i}.$$

Let $\Theta_{f_0}(t,z) = \sum \theta_{\alpha\beta}\, t^\alpha z^\beta, \theta_{\alpha,\beta}\in\mathbb{K}$, be the decomposition of the Bezoutian. We order the monomials that appear in Θ_{f_0}. The Bezoutian matrix of $f_0,\dots,f_n$ is the matrix $B_{f_0,\dots,f_n} = (\theta_{\alpha\beta})_{\alpha,\beta}$ (also simply denoted by B_{f_0} if $f_1,\dots,f_n$ are fixed).

The Bezoutian was used by E. Bézout to construct the resultant of two polynomials in one variable (see (Bézout, 1764)).

Definition 3.2. *Let $\mathbf{v} = (v_i)_{i\in\mathbb{N}}, \mathbf{w} = (w_j)_{j\in\mathbb{N}}$ be two $\mathbb{K}$-bases of R, and let*

$$\Theta_{f_0} = \sum_{i,j} \nu_{ij}\, v_i \otimes w_j \ , \quad \nu_{ij}\in\mathbb{K} \ ,$$

be the decomposition of the Bezoutian in these bases. The coefficient matrix $(\nu_{ij})_{i,j}$ will be denoted by $B^{\mathbf{v},\mathbf{w}}_{f_0}$.

If $\mathbf{v} = \mathbf{w} = (t^\alpha)_{\alpha\in\mathbb{N}^n}$, then $B^{\mathbf{v},\mathbf{w}}_{f_0}$ is the Bezoutian matrix of $f_0,\dots,f_n$. The matrix $B^{\mathbf{v},\mathbf{w}}_{f_0}$ is exactly the matrix of the $\mathbb{K}$-linear map

$$\begin{array}{rcl} \Theta^{\triangleright}_{f_0} : \widehat{R} & \to & R \\ \Lambda & \mapsto & \Theta^{\triangleright}_{f_0}(\Lambda) := \sum_{i,j} \nu_{ij}\Lambda(w_j)v_i \end{array}$$

in the dual basis $(\widehat{w_j})_{j\in\mathbb{N}}$ [†] of $\widehat{R}$ and the basis $(v_i)_{i\in\mathbb{N}}$ of R.

Similarly, we define $\Theta^{\triangleleft}_{f_0}(\Lambda) := \sum \nu_{ij}\,\Lambda(v_i)w_j, \Lambda\in\widehat{R}$. The matrix of this map in the bases $(\widehat{v_j})_{j\in\mathbb{N}}$ and $(w_i)_{i\in\mathbb{N}}$ is the transpose of $B^{\mathbf{v},\mathbf{w}}_{f_0}$.

The following proposition shows that the Bezoutian matrices B_{f_0}, for all $f_0\in R$, admit a diagonal decomposition in a common basis. It will be used in the following subsection.

Proposition 3.3. *Let $I = (f_1,\dots,f_n)$ be an ideal of R such that the $\mathbb{K}$-vector space $\mathcal{A} = R/I$ is of finite dimension D. Then there exists two bases $\mathbf{v} = (v_i)_{i\in\mathbb{N}}$ and $\mathbf{w} = (w_i)_{i\in\mathbb{N}}$ of R such that $(\overline{v}_1,\dots,\overline{v}_D), (\overline{w}_1,\dots,\overline{w}_D)$ are bases of $\mathcal{A}$, $v_i, w_i\in I$ for $i > D$,*

[†] The dual basis satisfy $\widehat{w_j}(w_i) = 1$ iff $i = j$ and 0 otherwise.

and for any f_0 in R the matrix $B_{f_0}^{\mathbf{v},\mathbf{w}}$ is of the form

$$\begin{array}{cc} \begin{array}{cc} v_1 \, \dots \, v_D & v_{D+1} \dots \end{array} & \\ \left(\begin{array}{c|c} M_{f_0} & \mathbf{0} \\ \hline \mathbf{0} & L_{f_0} \end{array}\right) & \begin{array}{c} w_1 \\ \vdots \\ w_D \\ w_{D+1} \\ \vdots \end{array} \end{array} \tag{3.1}$$

where M_{f_0} is the matrix of multiplication by f_0 in the basis $(\overline{v}_1, \dots, \overline{v}_D)$ of $\mathcal{A}$.

Proof. We recall that $\widehat{\mathcal{A}}$ is identified with $I^\perp$. Let us consider the two vector subspaces $E = \Theta_1^\triangleright(\widehat{\mathcal{A}})$ and $F = \Theta_1^\triangleleft(\widehat{\mathcal{A}})$ of R. From $\dim_{\mathbb{K}}(\widehat{\mathcal{A}}) = D$, we deduce that E and F are of dimension $\leq D$. According to (Kunz, 1986), (Scheja and Storch, 1975), $\Theta_1^\triangleright$ and $\Theta_1^\triangleleft$ are isomorphisms between $\widehat{\mathcal{A}}$ and $\mathcal{A}$. Therefore, the image of $\widehat{\mathcal{A}}$ by $\Theta_1^\triangleright$ and $\Theta_1^\triangleleft$ are at least of dimension D. Consequently, $\dim E = \dim F = D$ and E is isomorphic as a vector space to $\mathcal{A}$. Thus we have $R = E \oplus I$ and by symmetry $R = F \oplus I$.

From this, we deduce that Θ_1 is in $E \otimes F \oplus I \otimes I$, for it is in $E \otimes F \oplus E \otimes I \oplus I \otimes F \oplus I \otimes I$ and $\Theta_1^\triangleright(I^\perp) = E$, $\Theta_1^\triangleleft(I^\perp) = F$.

Let us fix now f_0 in R. It is clear from the definition 3.1 and from the invariance of the Bezoutian when we substitute z for t in the first column, that $\Theta_{f_0} - (1 \otimes f_0)\Theta_1$ is in the ideal of $R \otimes_{\mathbb{K}} R$ generated by $1 \otimes f_1, \dots, 1 \otimes f_n$. Consequently,

$$\Theta_{f_0}^\triangleright(\widehat{\mathcal{A}}) = \big((1 \otimes f_0)\,\Theta_1\big)^\triangleright(\widehat{\mathcal{A}}) = \Theta_1^\triangleright(f_0 \cdot \widehat{\mathcal{A}}) \subset \Theta_1^\triangleright(\widehat{\mathcal{A}}) = E.$$

The same argument shows that $\Theta_{f_0}^\triangleleft(\widehat{\mathcal{A}}) \subset F$, and $\Theta_{f_0} \in E \otimes F \oplus I \otimes I$.

Let $\mathbf{v} = (v_i)_{i \in \mathbb{N}}$ and $\mathbf{w} = (w_i)_{i \in \mathbb{N}}$ be two bases of R such that $(v_1, \dots, v_D)$ is a basis of E, $(w_1, \dots, w_D)$ a basis of F and $v_i \in I$, $w_i \in I$ for $i > D$. From the decomposition $\Theta_{f_0} \in E \otimes F \oplus I \otimes I$, $B_{f_0}^{\mathbf{v},\mathbf{w}}$ has a block-diagonal form.

Let us denote by $C_{f_0} = \big(c_{ij}(f_0)\big)_{i,j=1,\dots,D}$ the upper-left block in this decomposition and by $M_{f_0} = (m_{ij})_{i,j=1,\dots,D}$ the matrix of multiplication by f_0 in the basis $(\overline{v}_1, \dots, \overline{v}_D)$ of $\mathcal{A}$. We deduce from the decomposition above that, modulo the ideal $\big(f_1 \otimes 1, \dots, f_n \otimes 1\big)$, we have

$$\begin{aligned} \Theta_{f_0} \;&\equiv\; \sum_{i,j=1}^{D} c_{ij}(f_0)\, v_i \otimes w_j \;\equiv\; (f_0 \otimes 1)\,\Theta_1 \;\equiv\; (f_0 \otimes 1) \left(\sum_{i,j=1}^{D} c_{ij}(1)\, v_i \otimes w_j \right) \\ &\equiv\; \sum_{i,j=1}^{D} c_{ij}(1)\,(f_0\; v_i) \otimes w_j \;\equiv\; \sum_{k,j=1}^{D} \left(\sum_{i=1}^{D} m_{ki} c_{ij}(1) \right) v_k \otimes w_j \;, \end{aligned}$$

which implies that $C_{f_0} = M_{f_0}\, C_1$.

Notice that the matrix C_1 is invertible, for it is the matrix of $\Theta_1^\triangleright$ in the bases $(\overline{v}_1, \dots, \overline{v}_D)$ of $\mathcal{A}$ and its dual basis in $\widehat{\mathcal{A}}$. Indeed, as $f_1, \dots, f_n$ is a complete intersection, this map is an isomorphism between $\widehat{\mathcal{A}}$ and $\mathcal{A}$ (see (Scheja and Storch, 1975), (Kunz, 1986), (Becker *et al.*, 1996), (Elkadi and Mourrain, 1996)). By a change of bases, we may assume that $C_1 = \mathbb{I}_{\mathbb{D}}$ (the matrix identity), so that the matrix of $B_{f_0}^{\mathbf{v},\mathbf{w}}$ is of the form (3.1). $\square$

3.2. Bezoutians and Resultants

We consider here the system (2.1) of $n+1$ "polynomials" in n variables.

Theorem 3.4. *Assume that the conditions* **(D)** *are satisfied. Then any maximal minor of the Bezoutian matrix $B_{f_0,\dots,f_n}$ is divisible by the resultant* $\mathrm{Res}_X(\mathbf{f_c})$.

Proof. According to the conditions **(D)**, the set of coefficients $(c_{i,j})$ of $f_1,\dots,f_n$ such that $\mathcal{V}(f_1,\dots,f_n)$ is finite is a dense subset of $\mathbb{P}^{k_1}\times\cdots\times\mathbb{P}^{k_n}$. As $X^o=\sigma(U)$ is a dense subset of X, the set of $(c_{i,j})$ such that $\mathcal{V}(f_1,\dots,f_n)$ is finite and in X^o is also a dense subset. Let us choose "generic" coefficients in this subset for $f_1,\dots,f_n$.

The $\mathbb{K}$-vector space $R/(f_1,\dots,f_n)$ is of finite dimension. Let us denote by D_g the generic dimension of this quotient. For any $f_0\in R$, we denote by $r_g(f_0)$ the generic rank of the Bezoutian matrix B_{f_0}. The minors of size $r_g(f_0)$ of B_{f_0} are polynomials in $\mathbf{c}$, which are not all identically zero and any minor of size $r_g(f_0)+1$ vanishes identically.

According to proposition 3.3, for generic values of $\mathbf{c}$, the matrix B_{f_0} can be decomposed as in (3.1), so that

$$\mathrm{rank}(B_{f_0})=\mathrm{rank}(M_{f_0})+\mathrm{rank}(L_{f_0}).$$

As for generic values of $\mathbf{c}$, the variety $\mathcal{V}(f_0,\dots,f_n)$ is empty, the multiplication matrix M_{f_0} is generically invertible (the eigenvalues of M_{f_0} are the values of f_0 at the roots of $f_1,\dots,f_n$), that is of rank $D_g=\dim_{\mathbb{K}}\left(R/(f_1,\dots,f_n)\right)$.

Let us choose now $f_1,\dots,f_n$ such that their roots are in X^o and f_0 has a common root with $f_1,\dots,f_n$. In this case, $\mathrm{Res}_X(\mathbf{f_c})=0$. Moreover, we have $\mathrm{rank}(M_{f_0})<D_g$ (for f_0 vanishes at one of the roots of $f_1,\dots,f_n$), and by specialization the rank of L_{f_0} cannot exceed the generic one. Thus, $\mathrm{rank}(B_{f_0})<r_g(f_0)$ and all the $r_g(f_0)\times r_g(f_0)$ minors vanish.

As the set of systems $(f_0,\dots,f_n)$ such that $\mathcal{V}(f_1,\dots,f_n)\subset X^o$ and f_0 vanishes at one of these points is a dense subset of the variety $\mathcal{V}\big(\mathrm{Res}_X(\mathbf{f_c})\big)$ in $\mathbb{P}^{k_0}\times\ldots\times\mathbb{P}^{k_n}$, it implies that any maximal minor of the Bezoutian matrix vanishes on this resultant variety. Consequently, any maximal minor (of size $r_g(f_0)$) is divisible by the resultant. □

4. Examples and applications

We illustrate now our methods by some experiments in maple. It should be noticed that the eliminant polynomials are very large and cannot be computed with classical elimination methods like Gröbner basis techniques.

4.1. An example where the classical and toric resultants are degenerate

Consider the three following polynomials:

$$\begin{cases} f_0=c_{0,0}+c_{0,1}t_1+c_{0,2}t_2+c_{0,3}({t_1}^2+{t_2}^2)\\ f_1=c_{1,0}+c_{1,1}t_1+c_{1,2}t_2+c_{1,3}({t_1}^2+{t_2}^2)+c_{1,4}({t_1}^2+{t_2}^2)^2\\ f_2=c_{2,0}+c_{2,1}t_1+c_{2,2}t_2+c_{2,3}({t_1}^2+{t_2}^2)+c_{2,4}({t_1}^2+{t_2}^2)^2.\end{cases}$$

We are looking for conditions on the coefficients $c_{i,j}$ such that these three polynomials have a common root in $\mathbb{A}^2$. The resultant of these polynomials over $\mathbb{P}^2$ is zero (whatever the values of $(c_{i,j})$, for the polynomials f_0,f_1,f_2 vanish at the points $(0:1:i)$ and

$(0:1:-i)$. For the same reason, the toric resultant of these polynomials also vanishes (these polynomials have common roots in the associated toric variety).

Now applying the result of the previous section, we consider the map

$$\begin{array}{rcl}\sigma:\mathbb{A}^2 & \to & \mathbb{P}^3\\ (t_1,t_2) & \mapsto & (1:t_1:t_2:t_1^2+t_2^2)\end{array}$$

whose Jacobian is of rank 2. We denote by X the closure of the image of σ in $\mathbb{P}^3$ and let

$$\begin{array}{rcl}\psi_0 & = & (x_0,x_1,x_2,x_3)\\ \psi_1 & = & (x_0^2,x_0x_1,x_0x_2,x_0x_3,x_3^2)\\ \psi_2 & = & (x_0^2,x_0x_1,x_0x_2,x_0x_3,x_3^2)\end{array}$$

where $(x_0:x_1:x_2:x_3)$ are the homogeneous coordinates of $\mathbb{P}^3$. We have the decomposition $f_i=\sum c_{i,j}\psi_{i,j}\circ\sigma$, $i=0,1,2$. For generic values of the coefficients $c_{i,j}$, we check that the system $f_1=f_2=0$ has a finite number of solutions in $\mathbb{A}^2$, and so that by theorem 3.4, any nonzero maximal minor of the Bezoutian matrix B_{f_0,f_1,f_2} is divisible by the resultant $\mathrm{Res}_{\mathrm{X}}(f_0,f_1,f_2)$. Computing a maximal minor of this Bezoutian matrix of size 12×12, and rank 10, yields a huge polynomial in the coefficients $c_{i,j}$, containing 207805 monomials. It can be factorized as $Q_1Q_2(Q_3)^2S$, where

$$\begin{array}{rcl}Q_1 & = & -c_{0,2}c_{1,3}c_{2,4}+c_{0,2}c_{1,4}c_{2,3}+c_{1,2}c_{0,3}c_{2,4}-c_{2,2}c_{0,3}c_{1,4}\\ Q_2 & = & c_{0,1}c_{1,3}c_{2,4}-c_{0,1}c_{1,4}c_{2,3}-c_{1,1}c_{0,3}c_{2,4}+c_{2,1}c_{0,3}c_{1,4}\\ Q_3 & = & {c_{0,3}}^2{c_{1,1}}^2{c_{2,4}}^2-2{c_{0,3}}^2c_{1,1}c_{2,1}c_{2,4}c_{1,4}+{c_{0,3}}^2{c_{2,4}}^2{c_{1,2}}^2+\cdots\\ S & = & {c_{2,0}}^4{c_{1,4}}^4{c_{0,2}}^4+{c_{2,0}}^4{c_{1,4}}^4{c_{0,1}}^4+{c_{1,0}}^4{c_{2,4}}^4{c_{0,2}}^4+{c_{1,0}}^4{c_{2,4}}^4{c_{0,1}}^4+\cdots\end{array}$$

The polynomials Q_3 and S contain respectively 20 and 2495 monomials. As for generic equations f_0,f_1,f_2, the number of points in $\mathcal{V}(f_0,f_1)$, $\mathcal{V}(f_0,f_2)$, $\mathcal{V}(f_1,f_2)$ is 4 (see for instance (Mourrain, 1996)), according to remark 2.5 $\mathrm{Res}_X(f_0,f_1,f_2)$ is homogeneous of degree 4 in the coefficients of each polynomial f_i. Thus, the resultant $\mathrm{Res}_X(\mathbf{f})$ corresponds to the last factor S.

4.2. An example of the resultant as an implicit equation

We want to compute the “resultant” of the system

$$\left\{\begin{array}{l}f_0=c_{0,0}t_1t_2+c_{0,1}{t_2}^2+c_{0,2}t_3+c_{0,3}\left({t_1}^3+{t_2}^3+{t_3}^3\right)+c_{0,4}\left(t_1{t_2}^2+{t_1}^2t_2-{t_3}^3\right)\\ f_1=c_{1,0}t_1t_2+c_{1,1}{t_2}^2+c_{1,2}t_3+c_{1,3}\left({t_1}^3+{t_2}^3+{t_3}^3\right)+c_{1,4}\left(t_1{t_2}^2+{t_1}^2t_2-{t_3}^3\right)\\ f_2=c_{2,0}t_1t_2+c_{2,1}{t_2}^2+c_{2,2}t_3+c_{2,3}\left({t_1}^3+{t_2}^3+{t_3}^3\right)+c_{2,4}\left(t_1{t_2}^2+{t_1}^2t_2-{t_3}^3\right)\\ f_3=c_{3,0}t_1t_2+c_{3,1}{t_2}^2+c_{3,2}t_3+c_{3,3}\left({t_1}^3+{t_2}^3+{t_3}^3\right)+c_{3,4}\left(t_1{t_2}^2+{t_1}^2t_2-{t_3}^3\right)\end{array}\right.$$

Following the previous sections, we consider the map:

$$\begin{array}{rcl}\sigma:\mathbb{A}^3-\{(0,0,0)\} & \to & \mathbb{P}^4\\ (t_1,t_2,t_3) & \mapsto & (t_1t_2:t_2^2:t_3:{t_1}^3+{t_2}^3+{t_3}^3:t_1{t_2}^2+{t_1}^2t_2-{t_3}^3)\end{array}$$

whose Jacobian is generically of rank 3. We denote by X the closure of the image of σ in $\mathbb{P}^4$. We decompose as $f_i=\sum c_{i,j}\psi_{i,j}\circ\sigma$ with

$$\psi_i \;=\; (x_0,x_1,x_2,x_3,x_4),\quad i=0,1,2,3.$$

We check that the system $f_1 = f_2 = f_3 = 0$ has a finite number of solutions in $\mathbb{A}^3 - \{(0,0,0)\}$ for generic values of $\mathbf{c}$. Thus according to theorem 3.4, any maximal minor of the Bezoutian matrix B_{f_0,f_1,f_2,f_3} is divisible by the resultant $\mathrm{Res}_X(f_0, f_1, f_2, f_3)$.

Computing a maximal minor of this Bezoutian matrix, which is a 25×23 matrix of rank 21, we obtain

$$\begin{aligned}
&\Delta_5 \Delta_2 {\Delta_3}^6 \\
&\big({\Delta_5}^4 {\Delta_3}^3 {\Delta_2}^6 - 4\,{\Delta_5}^3 \Delta_4 {\Delta_3}^3 {\Delta_2}^6 + 6\,{\Delta_5}^2 {\Delta_4}^2 {\Delta_3}^3 {\Delta_2}^6 - 4\,\Delta_5 {\Delta_4}^3 {\Delta_3}^3 {\Delta_2}^6 + \Delta_5 {\Delta_2}^{12} \\
&-3\,\Delta_5 {\Delta_2}^{11} \Delta_1 + 6\,\Delta_5 {\Delta_2}^{10} {\Delta_1}^2 - 11\,\Delta_5 {\Delta_2}^9 {\Delta_1}^3 + 15\,\Delta_5 {\Delta_2}^8 {\Delta_1}^4 - 18\,\Delta_5 {\Delta_2}^7 {\Delta_1}^5 \\
&+20\,\Delta_5 {\Delta_2}^6 {\Delta_1}^6 - 18\,\Delta_5 {\Delta_2}^5 {\Delta_1}^7 + 15\,\Delta_5 {\Delta_2}^4 {\Delta_1}^8 - 11\,\Delta_5 {\Delta_2}^3 {\Delta_1}^9 \\
&+6\,\Delta_5 {\Delta_2}^2 {\Delta_1}^{10} - 3\,\Delta_5 \Delta_2 {\Delta_1}^{11} + \Delta_5 {\Delta_1}^{12} + {\Delta_4}^4 {\Delta_3}^3 {\Delta_2}^6 - \Delta_4 {\Delta_2}^{11} \Delta_1 + 4\,\Delta_4 {\Delta_2}^{10} {\Delta_1}^2 \\
&-9\,\Delta_4 {\Delta_2}^9 {\Delta_1}^3 + 16\,\Delta_4 {\Delta_2}^8 {\Delta_1}^4 - 22\,\Delta_4 {\Delta_2}^7 {\Delta_1}^5 + 24\,\Delta_4 {\Delta_2}^6 {\Delta_1}^6 - 22\,\Delta_4 {\Delta_2}^5 {\Delta_1}^7 \\
&+16\,\Delta_4 {\Delta_2}^4 {\Delta_1}^8 - 9\,\Delta_4 {\Delta_2}^3 {\Delta_1}^9 + 4\,\Delta_4 {\Delta_2}^2 {\Delta_1}^{10} - \Delta_4 \Delta_2 {\Delta_1}^{11}\big)
\end{aligned}$$

where Δ_i is the determinant of the submatrix of

$$C = \begin{bmatrix} c_{0,0} & c_{0,1} & c_{0,2} & c_{0,3} & c_{0,4} \\ c_{1,0} & c_{1,1} & c_{1,2} & c_{1,3} & c_{1,4} \\ c_{2,0} & c_{2,1} & c_{2,2} & c_{2,3} & c_{2,4} \\ c_{3,0} & c_{3,1} & c_{3,2} & c_{3,3} & c_{3,4} \end{bmatrix}$$

corresponding to all columns except i. In fact the coefficients of B_{f_0,f_1,f_2,f_3} are linear forms in Δ_i, $i \in \{1, 2, 3, 4, 5\}$. The last factor of this expansion is an irreducible polynomial in $c_{i,j}$ and corresponds to the resultant of $\mathbf{f_c}$ over X. Another way to look at this polynomial is to consider it as the condition on the coefficients $\mathbf{c}$ such that the projective point in the kernel of C (whose coordinates are $(\Delta_1 : \cdots : \pm\Delta_5)$) lies on the closure of the image of σ. Thus replacing Δ_i by the variable $\pm x_i$, we obtain the implicit equation of degree 13 of the image of σ. When $f_0, \ldots, f_n$ have the same support, this computation involving the $n+1 \times n+1$ determinants of the coefficients $c_{i,j}$ of the input polynomials can easily be generalized, using the multilinearity and antisymmetry of the Bezoutian matrix.

4.3. Blowing up and resultants

A typical situation, which appears in many practical cases is when we have to deal with an overconstrained system of $n+1$ homogeneous equations in $X = \mathbb{P}^n$ depending on parameters $\mathbf{c}$, but whose zero set contains a variety Y independent of these parameters. In such a case, each equation f_i can be decomposed as

$$f_i(t) = \sum_{j=0}^{m} q_{i,j}(\mathbf{c}, t)\; p_j(t) \quad , \quad i = 0, \ldots, n \ ,$$

where $q_{i,j}(\mathbf{c}, t)$ are polynomials in t and $\mathbf{c}$, and where the polynomials $p_j(t)$, $j = 0, \ldots, m$ define the variety Y. We are looking for the conditions on the parameters $\mathbf{c}$ such that this system has a root outside Y, also called residual intersection conditions.

A standard construction used to analyze what happens outside Y consists in blowing up X along Y, by computing the closure of the graph of the rational map

$$\begin{aligned} \phi : X - Y &\ \rightarrow\ \mathbb{P}^m \\ x &\ \mapsto\ \big(p_0(x) : \cdots : p_m(x)\big) \end{aligned}$$

in $X \times \mathbb{P}^m$. The blowing up of X along Y is denoted by $\tilde{X}_Y$.

Notice that the closure of the graph of the restriction of ϕ on any dense subset of $X - Y$ is also $\tilde{X}_Y$ (see (Harris, 1992)[p. 82]).

Let $X^o = \mathbb{A}^n - Y$ be the set of points $(1 : t_1 : \cdots : t_n)$ of $X = \mathbb{P}^n$ and let us define the map

$$\begin{array}{rcl} \sigma : \mathbb{A}^n - Y & \to & X \times \mathbb{P}^m \\ (t_1, \ldots, t_n) & \mapsto & ((1 : t_1 : \ldots : t_n), \big(p_0(1 : t_1 : \ldots : t_n) : \cdots : p_m(1 : t_1 : \ldots : t_n))\big). \end{array}$$

We immediately check that the rank of the Jacobian of this map is generically n. The closure of the image of σ is the strict transform $\tilde{X}_Y$ of the blow up of X along Y. If we assume moreover that for generic values of the parameters $\mathbf{c}$ the system $f_1 = \cdots = f_n = 0$ has a finite number of solutions in $\mathbb{A}^n - Y$, then according to theorem 2.3, the resultant over $\tilde{X}_Y$ is well defined. Let us illustrate it by the following system

$$\left\{ \begin{array}{rcl} f_0 & = & c_{0,0} + c_{0,1}t_1 + c_{0,2}t_2 + c_{0,3}t_3 + c_{0,4}t_4 + c_{0,5}\left(t_1{}^2 - t_2t_3\right) + c_{0,6}\left(t_1t_2 - t_4{}^2\right) \\ f_1 & = & c_{1,0} + c_{1,1}t_1 + c_{1,2}t_2 + c_{1,3}t_3 + c_{1,4}t_4 + c_{1,5}\left(t_1{}^2 - t_2t_3\right) + c_{1,6}\left(t_1t_2 - t_4{}^2\right) \\ f_2 & = & c_{2,0} + c_{2,1}t_1 + c_{2,2}t_2 + c_{2,3}t_3 + c_{2,4}t_4 + c_{2,5}\left(t_1{}^2 - t_2t_3\right) + c_{2,6}\left(t_1t_2 - t_4{}^2\right) \\ f_3 & = & c_{3,0} + c_{3,1}t_1 + c_{3,2}t_2 + c_{3,3}t_3 + c_{3,4}t_4 + c_{3,5}\left(t_1{}^2 - t_2t_3\right) + c_{3,6}\left(t_1t_2 - t_4{}^2\right) \\ f_4 & = & c_{4,0} + c_{4,1}t_1 + c_{4,2}t_2 + c_{4,3}t_3 + c_{4,4}t_4 + c_{4,5}\left(t_1{}^2 - t_2t_3\right) + c_{4,6}\left(t_1t_2 - t_4{}^2\right) \end{array} \right.$$

For generic values of the parameters, we check that the system $f_1 = f_2 = f_3 = f_4 = 0$ has a finite number of roots (i.e. 5).

After homogenization $t_i = \frac{x_i}{x_0}$, we see that the zero set of every homogenized equation contains the twisted cubic $\mathcal{C}$ defined by $x_0 = 0, x_2x_4 - x_3{}^2 = 0, x_1x_4 - x_3x_2 = 0, x_1x_3 - x_2{}^2 = 0$ in the plane at infinity. We blow up this curve by considering the closure of the graph of

$$\begin{array}{rcl} \phi : \mathbb{P}^4 - \mathcal{C} & \to & \mathbb{P}^3 \\ (x_0 : x_1 : x_2 : x_3 : x_4) & \mapsto & (x_0{}^2 : x_2x_4 - x_3{}^2 : x_1x_4 - x_3x_2 : x_1x_3 - x_2{}^2). \end{array}$$

The map σ that we use to parameterize a dense open subset of this graph is

$$\begin{array}{rcl} \sigma : \mathbb{A}^4 & \to & \mathbb{P}^4 \times \mathbb{P}^3 \\ (t_1, t_2, t_3, t_4) & \mapsto & \big((1 : t_1 : t_2 : t_3 : t_4), (1 : t_2t_4 - t_3{}^2 : t_1t_4 - t_3t_2 : t_1t_3 - t_2{}^2)\big). \end{array}$$

Its Jacobian is of rank 4. A maximal minor of the Bezoutian matrix $B_{f_0,\ldots,f_n}$, expressed in terms of the 5×5 determinants $\Delta_{i,j}$, $i, j \in \{1, 2, 3, 4, 5, 6, 7\}$ of the coefficient matrix $(c_{i,j})_{0 \le i \le 4,\, 0 \le j \le 6}$ corresponding to all columns except i and j is

$$-\Delta_{1,2}\Delta_{4,5}\Delta_{1,3}\Delta_{3,5}\Delta_{1,4}\Delta_{2,5}\Delta_{1,7} + \Delta_{1,3}{}^2\Delta_{1,2}\Delta_{4,5}{}^2\Delta_{2,3}\Delta_{2,5} + \Delta_{1,4}{}^2\Delta_{1,3}\Delta_{2,3}\Delta_{1,7}{}^2\Delta_{2,5} + \cdots$$

The resultant over the blowing up of $\mathbb{P}^4$ along $\mathcal{C}$ is this polynomial in $c_{i,j}$ divided by $\Delta_{2,3,5,6}\,\Delta_{1,5}$. We should remark that this resultant vanishes not only when the hypersurfaces $\mathcal{V}(f_i)$ intersect outside $\mathcal{C}$, but also when they share a common tangent along $\mathcal{C}$.

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